Homework 1 Solutions – Relativity (`G0I36A`)

18/11/2020

1 The Three-Sphere

1.) (3 points) In matrix form, with entries indexed by (θ, ϕ, ψ), the metric is

$$g_{\mu\nu} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix} \ . \tag{1.1}$$

Inverting this matrix yields

$$g^{\mu\nu} = 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sin^2\theta} & -\frac{\cos\theta}{\sin\theta} \\ 0 & -\frac{\cos\theta}{\sin^2\theta} & \frac{1}{\sin^2\theta} \end{pmatrix} \ . \tag{1.2}$$

Since the metric components depend only on θ, all non-zero Christoffel symbols must have at least one index equal to θ. Moreover, since $g_{\theta\theta}$ is constant and $g^{\theta\mu}$ and $g_{\theta\mu}$ are both non-zero only when $\mu = \theta$, any symbol with two or three instances of θ will automatically be zero.

Additionally, symbols like $\Gamma^{\theta}_{\phi\phi}$ and $\Gamma^{\theta}_{\psi\psi}$ vanish automatically, since $g_{\phi\phi}$ and $g_{\psi\psi}$ are both constant and thus independent of θ. The Christoffel symbols are also symmetric in their lower indices. Finally, the metric is invariant under $\phi \leftrightarrow \psi$, and so the Christoffel symbols will be as well. This leaves us with only three independent, non-zero symbols that we must calculate:

$$\Gamma^{\theta}_{\phi\psi} \ , \ \Gamma^{\phi}_{\theta\phi} \ , \ \Gamma^{\phi}_{\theta\psi} \ . \tag{1.3}$$

The formula for the Christoffel symbols is

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \ . \tag{1.4}$$

Plugging in for the three symbols above and using our symmetries yields the following non-zero symbols:

$$\begin{aligned} \Gamma^{\theta}_{\phi\psi} &= \Gamma^{\theta}_{\psi\phi} = -\frac{1}{2} g^{\theta\theta}\partial_\theta g_{\phi\psi} = \frac{\sin\theta}{2} \ , \\ \Gamma^{\phi}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} = \Gamma^{\psi}_{\theta\psi} = \Gamma^{\psi}_{\psi\theta} = \frac{1}{2} g^{\phi\psi}\partial_\theta g_{\phi\psi} = \frac{\cos\theta}{2\sin\theta} \ , \\ \Gamma^{\phi}_{\theta\psi} &= \Gamma^{\phi}_{\psi\theta} = \Gamma^{\psi}_{\theta\phi} = \Gamma^{\psi}_{\phi\theta} = \frac{1}{2} g^{\phi\phi}\partial_\theta g_{\phi\psi} = -\frac{1}{2\sin\theta} \ . \end{aligned} \tag{1.5}$$

2.) (4 points) The Riemann tensor $R_{\mu\nu\rho\sigma}$ has the following (anti-)symmetry properties:

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu} \ . \tag{1.6}$$

The antisymmetry in the first and second pairs of indices mean that the Riemann tensor vanishes whenever an index appears more than twice. Then, accounting for the exchange symmetry of the first and second pairs, we conclude that the following six components are independent and not automatically zero:

$$R_{\theta\phi\theta\phi}\ ,\ R_{\theta\psi\theta\psi}\ ,\ R_{\phi\psi\phi\psi}\ ,\ R_{\theta\phi\theta\psi}\ ,\ R_{\theta\phi\psi\phi}\ ,\ R_{\theta\psi\phi\psi}\ . \tag{1.7}$$

But, if we now account for the $\phi \leftrightarrow \psi$ symmetry of the metric, we end up with only four independent components we must compute:

$$R_{\theta\phi\theta\phi}\left(=R_{\theta\psi\theta\psi}\right)\ ,\ R_{\phi\psi\phi\psi}\ ,\ R_{\theta\phi\theta\psi}\ ,\ R_{\theta\phi\psi\phi}\left(=R_{\theta\psi\phi\psi}\right)\ . \tag{1.8}$$

To compute these, we need to first calculate the relevant components of the $(1,3)$ version of the Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma}=\partial_\rho\Gamma^{\mu}_{\nu\sigma}-\partial_\sigma\Gamma^{\mu}_{\nu\rho}+\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\rho}\ , \tag{1.9}$$

and then lower indices appropriately. We compute that

$$\begin{aligned}
R^{\theta}{}_{\phi\theta\phi}&=-\Gamma^{\theta}_{\phi\psi}\Gamma^{\psi}_{\phi\theta}=\frac{1}{4}\ ,\\
R^{\theta}{}_{\phi\theta\psi}&=\partial_\theta\Gamma^{\theta}_{\phi\psi}-\Gamma^{\theta}_{\psi\phi}\Gamma^{\phi}_{\phi\theta}=\frac{\cos\theta}{4}\ ,\\
R^{\theta}{}_{\phi\psi\phi}&=\Gamma^{\theta}_{\psi\lambda}\Gamma^{\lambda}_{\phi\phi}-\Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\psi}=0\ ,\\
R^{\phi}{}_{\psi\phi\psi}&=-\Gamma^{\phi}_{\psi\theta}\Gamma^{\theta}_{\psi\phi}=\frac{1}{4}\ ,\\
R^{\psi}{}_{\psi\phi\psi}&=-\Gamma^{\psi}_{\psi\theta}\Gamma^{\theta}_{\psi\phi}=-\frac{\cos\theta}{4}\ .
\end{aligned} \tag{1.10}$$

With these, the four independent components (1.8) of the Riemann tensor are

$$\begin{aligned}
R_{\theta\phi\theta\phi}&=g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi}=\frac{1}{16}\ ,\\
R_{\phi\psi\phi\psi}&=g_{\phi\phi}R^{\phi}{}_{\psi\phi\psi}+g_{\phi\psi}R^{\psi}{}_{\psi\phi\psi}=\frac{\sin^2\theta}{16}\ ,\\
R_{\theta\phi\theta\psi}&=g_{\theta\theta}R^{\theta}{}_{\phi\theta\psi}=\frac{\cos\theta}{16}\ ,\\
R_{\theta\phi\psi\phi}&=g_{\theta\theta}R^{\theta}{}_{\phi\psi\phi}=0\ .
\end{aligned} \tag{1.11}$$

Now, we want to compute the Ricci tensor $R_{\mu\nu}=g^{\rho\sigma}R_{\rho\mu\sigma\nu}$. Since the Ricci tensor is symmetric, and since we still have the $\phi \leftrightarrow \psi$ symmetry, there are only four independent components that we need to compute:

$$R_{\theta\theta}\ ,\ R_{\phi\phi}\left(=R_{\psi\psi}\right)\ ,\ R_{\theta\phi}\left(=R_{\phi\theta}=R_{\theta\psi}=R_{\psi\theta}\right)\ ,\ R_{\phi\psi}\left(=R_{\psi\phi}\right)\ . \tag{1.12}$$

Using our independent Riemann tensor components, we compute these to be as follows:

$$\begin{aligned}
R_{\theta\theta}&=g^{\phi\phi}R_{\phi\theta\phi\theta}+g^{\psi\psi}R_{\psi\theta\psi\theta}+2g^{\phi\psi}R_{\phi\theta\psi\theta}=\frac{1}{2}\ ,\\
R_{\phi\phi}&=g^{\theta\theta}R_{\theta\phi\theta\phi}+g^{\psi\psi}R_{\psi\phi\psi\phi}=\frac{1}{2}\ ,\\
R_{\theta\phi}&=g^{\psi\psi}R_{\psi\theta\psi\phi}+g^{\phi\psi}R_{\phi\theta\psi\phi}=0\ ,\\
R_{\phi\psi}&=g^{\theta\theta}R_{\theta\phi\theta\psi}+g^{\psi\phi}R_{\psi\phi\phi\psi}=\frac{\cos\theta}{2}\ .
\end{aligned} \tag{1.13}$$

Finally, we need to compute the Ricci scalar $R=g^{\mu\nu}R_{\mu\nu}$. Note that since $g^{\mu\nu}$ has off-diagonal elements, the Ricci scalar will also involve the off-diagonal elements of $R_{\mu\nu}$. We find that

$$R=g^{\mu\nu}R_{\mu\nu}=g^{\theta\theta}R_{\theta\theta}+g^{\phi\phi}R_{\phi\phi}+g^{\psi\psi}R_{\psi\psi}+2g^{\phi\psi}R_{\phi\psi}=6\ , \tag{1.14}$$

where we ignored terms that depend on $R_{\theta\phi}$, since it is zero.

3.) (2 points) First, let us define

$$G_{\mu\nu\rho\sigma} \equiv g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \ . \tag{1.15}$$

Notice that this tensor satisfies the following (anti-)symmetry properties:

$$G_{\mu\nu\rho\sigma} = -G_{\nu\mu\rho\sigma} = -G_{\mu\nu\sigma\rho} = G_{\rho\sigma\mu\nu} \ . \tag{1.16}$$

These are precisely the same properties that the Riemann tensor obeys, and so both $R_{\mu\nu\rho\sigma}$ and $G_{\mu\nu\rho\sigma}$ can in principle be proportional. Since they obey the same symmetries, it suffices to compare only their few independent components to see if the tensors are proportional. Explicitly, we compute that

$$\begin{aligned}
G_{\theta\phi\theta\phi} &= g_{\theta\theta}g_{\phi\phi} = \frac{1}{16} \ , \\
G_{\phi\psi\phi\psi} &= g_{\phi\phi}g_{\psi\psi} - g_{\phi\psi}g_{\phi\psi} = \frac{\sin^2\theta}{16} \ , \\
G_{\theta\phi\theta\psi} &= g_{\theta\theta}g_{\phi\psi} = \frac{\cos\theta}{16} \ , \\
G_{\theta\phi\phi\psi} &= g_{\theta\phi}g_{\phi\psi} - g_{\theta\psi}g_{\phi\phi} = 0 \ .
\end{aligned} \tag{1.17}$$

Upon comparison with the four independent components (1.11) of the Riemann tensor, we can see that all independent components of $R_{\mu\nu\rho\sigma}$ and $G_{\mu\nu\rho\sigma}$ are exactly equal. Therefore, we conclude that

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \ , \tag{1.18}$$

and so $c_1 = 1$.

To show that $R_{\mu\nu} = c_2 g_{\mu\nu}$ for some constant c_2, we could in principle demonstrate that the symmetries match and then compare the independent components. An easier method is to simpy use our result (1.18) and contract with the metric tensor:

$$\begin{aligned}
R_{\mu\nu} &= g^{\rho\sigma}R_{\rho\mu\sigma\nu} \\
&= g^{\rho\sigma}(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma}) \\
&= \delta^\sigma_\sigma g_{\mu\nu} - \delta^\sigma_\nu g_{\mu\sigma} \\
&= 3g_{\mu\nu} - g_{\mu\nu} \\
&= 2g_{\mu\nu} \ .
\end{aligned} \tag{1.19}$$

Thus, $R_{\mu\nu} = c_2 g_{\mu\nu}$ with $c_2 = 2$. Note that we are considering a three-dimensional manifold, and thus $\delta^\mu_\mu = 3$, the number of spacetime dimensions.

4.) (1 point) First, we must compute the determinant g of the metric:

$$\begin{aligned}
g &= \det\left[\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix}\right] = \left(\frac{1}{4}\right)^3 \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix} \\
&= \left(\frac{1}{4}\right)^3 \times 1 \times \det\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} = \frac{1}{64}(1-\cos^2\theta) = \frac{\sin^2\theta}{64} \ .
\end{aligned} \tag{1.20}$$

Thus, the square root of the determinant is

$$\sqrt{g} = \frac{\sin\theta}{8} \ . \tag{1.21}$$

Taking into account the given integration ranges, the volume is

$$V = \int d^3x\,\sqrt{g} = \frac{1}{8}\int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \int_0^{\pi} \sin\theta = -\pi^2\left[\cos\theta\right]_{\theta=0}^{\theta=\pi} = 2\pi^2 \ . \tag{1.22}$$

2 Time Dilation for Orbiting Satellites

1.) (2 points) The satellite will follow a geodesic, and so its path $x^\mu(\lambda)$ is described by the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \ , \tag{2.1}$$

where λ is the affine parameter along the geodesic. We can take this to be τ, the proper time that the satellite experiences, and so we can write this as

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 \ , \tag{2.2}$$

where dots denote derivatives with respect to τ.

The satellite orbits at a constant r and θ, which means that both $\dot{r}$ and $\dot{\theta}$ are zero. The r-component of the geodesic equation therefore simplifies to

$$\Gamma^r_{tt}\dot{t}^2 + 2\Gamma^r_{t\phi}\dot{t}\dot{\phi} + \Gamma^r_{\phi\phi}\dot{\phi}^2 = 0 \ . \tag{2.3}$$

The three Christoffel symbols that appear in the above equation are the only Christoffel symbols we need to compute.

First, we note that the metric is diagonal, and so any symbol with three different indices (like $\Gamma^r_{t\phi}$) will vanish. Then, since the metric components depend only on r and θ, the two remaining symbols are computed as follows:

$$\begin{aligned}
\Gamma^r_{tt} &= -\frac{1}{2}g^{rr}\partial_r g_{tt} = \frac{1}{2(1-2\Phi)}\partial_r\left(c^2 - \frac{2GM}{r}\right) = \frac{GM}{r^2(1-2\Phi)} \ , \\
\Gamma^r_{\phi\phi} &= -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} = -\frac{1}{2(1-2\Phi)}\partial_r\left(r^2\sin^2\theta\right) = -\frac{r\sin^2\theta}{1-2\Phi} \ .
\end{aligned} \tag{2.4}$$

Note that we used the fact that the metric is diagonal to easily compute $g^{rr} = \frac{1}{g_{rr}}$. If we now fix $r = R$ and $\theta = \frac{\pi}{2}$ and plug the Christoffel symbols into the geodesic equation (2.3), we find that

$$\frac{1}{1-2\Phi}\left(\frac{GM}{R^2}\dot{t}^2 - r\dot{\phi}^2\right) = 0 \ , \tag{2.5}$$

the solution of which requires

$$\dot{t}^2 = \frac{R^3}{GM}\dot{\phi}^2 \ . \tag{2.6}$$

Note that we had to compute the Christoffel symbols *before* fixing r and θ to be constant. Doing this the other way around will give the wrong result.

2.) (3 points) In the remaining problems, let $R_S = R_E + R_o$ be the distance between the satellite and the center of the Earth. Our result from the previous problem tells us that any infinitesimal changes in time dt and angle $d\phi$ along the satellite's path are related by

$$dt^2 = \frac{R_S^3}{GM}d\phi^2 \ . \tag{2.7}$$

Additionally, since r and θ are fixed, $dr = d\theta = 0$ along the satellite's path. Therefore, the proper time τ that the satellite experiences is defined by

$$ds^2 = -c^2d\tau^2 = -\left(1 - \frac{2GM}{c^2R_S}\right)c^2dt^2 + R_S^2d\phi^2 \ . \tag{2.8}$$

Solving for $d\tau$, we find that

$$\begin{aligned} d\tau &= \sqrt{\left(1 - \frac{2GM}{c^2 R_S}\right) dt^2 - \frac{R_S^2}{c^2} d\phi^2} \\ &= \sqrt{\left(1 - \frac{2GM}{c^2 R_S}\right) \frac{R_S^3}{GM} d\phi^2 - \frac{R_S^2}{c^2} d\phi^2} \\ &= \sqrt{\frac{R_S^3}{GM} - \frac{3R_S^3}{c^2}}\, d\phi \ . \end{aligned} \tag{2.9}$$

For one full period, ϕ will go from $\phi = 0$ to $\phi = 2\pi$, and thus the change in proper time $\Delta\tau_S$ the satellite experiences over the course of this orbit is simply

$$\Delta\tau_S = \int_0^{2\pi} \sqrt{\frac{R_S^3}{GM} - \frac{3R_S^3}{c^2}}\, d\phi = 2\pi \sqrt{\frac{R_S^3}{GM} - \frac{3R_S^3}{c^2}} \ . \tag{2.10}$$

If we evaluate this explicitly, we find that

$$\Delta\tau_S \approx 43{,}102.943299 \text{ seconds} \ , \tag{2.11}$$

which is about 12 hours. Note that you might get a slightly different result depending on what values of G and M you use, which is totally fine.

3.) (3 points) To find the proper time an observer on Earth experiences while the satellite undergoes a full orbit, we need to first calculate the coordinate time Δt it takes for this orbit to occur. To compute this, we use the relationship (2.7) to express

$$dt = \sqrt{\frac{R_S^3}{GM}}\, d\phi \ , \tag{2.12}$$

and thus the coordinate time difference for one full period is

$$\Delta t = \int_0^{2\pi} \sqrt{\frac{R_S^3}{GM}}\, d\phi = 2\pi \sqrt{\frac{R_S^3}{GM}} \ . \tag{2.13}$$

Now, we need to relate this coordinate time to the proper time an observer on Earth experiences. Note that the Earth observer is not following a geodesic; they are standing on Earth (which we assume is not rotating), and thus r, θ, and ϕ are all constant for this observer. Thus the proper time of the observer is given by

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{c^2 R_E}\right) c^2 dt^2 \ . \tag{2.14}$$

Solving for $d\tau$, this yields

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 R_E}}\, dt \ . \tag{2.15}$$

We calculated the coordinate time difference Δt required for the satellite to make a full orbit in (2.13). If we integrate the above equation, we find that the proper time difference $\Delta\tau_E$ experienced on Earth during this orbit is

$$\Delta\tau_E = \int \sqrt{1 - \frac{2GM}{c^2 R_E}}\, dt = \sqrt{1 - \frac{2GM}{c^2 R_E}}\, \Delta t = 2\pi \sqrt{\frac{R_S^3}{GM} - \frac{2R_S^3}{c^2 R_E}} \ . \tag{2.16}$$

Importantly, this is *not* the same as the proper time experienced by the satellite! If we plug in numbers, this comes out to approximately

$$\Delta\tau_E \approx 43{,}102.943280 \text{ seconds} \ , \tag{2.17}$$

which is again close to 12 hours. The difference between the satellite and Earth proper times is

$$\Delta\tau_S - \Delta\tau_E \approx 1.9 \times 10^{-5} \text{ seconds} , \tag{2.18}$$

which is a very small (but non-zero) difference due to gravitational time dilation.

4.) (2 points) In the absence of any relativistic effects, the time ΔT it takes for the satellite to orbit is given simply by Kepler's Law:

$$\Delta T = 2\pi\sqrt{\frac{R_S^3}{GM}} . \tag{2.19}$$

Notice that this is the same as the coordinate time Δt that we calculated earlier. Additionally, we could have also reached this same conclusion if we took $c \to \infty$ in either our answer to problem 2 or to problem 3. We should also note that this is the *same* as the time the observer on Earth would feel as the satellite completes one orbit. Time dilation is a purely relativistic phenomenon, with no counterpart in classical mechanics.

If we plug in numbers, this non-relativistic period is given by

$$\Delta T \approx 43,102.943310 \text{ seconds} , \tag{2.20}$$

which is actually greater than both $\Delta\tau_S$ and $\Delta\tau_E$. To be more specific, the differences are

$$\begin{aligned} \Delta T - \Delta\tau_S &\approx 1.1 \times 10^{-5} \text{ seconds} , \\ \Delta T - \Delta\tau_E &\approx 3.0 \times 10^{-5} \text{ seconds} . \end{aligned} \tag{2.21}$$

So, not only do relativistic effects lead to a difference in the time felt between the observer on Earth and the satellite on the order of 10^{-5} seconds, but the relativistic effects *also* lead to a change in the proper orbital time itself, again on the order of 10^{-5} seconds.

These differences are small but non-negligible. If we were to use these satellites to triangulate someone's position on Earth by sending out light rays to the satellites, this error of $\Delta t \sim 10^{-5}$ seconds leads to an error in measuring distances on the order of

$$\Delta d \sim c\Delta t \sim 3 \text{ km per orbit} . \tag{2.22}$$

Over the course of just a single day, the satellite undergoes about two rotations, and so the error thus accumulates to about 6 km if we do not account for relativity correctly! It is therefore crucial for GPS navigation that we understand relativity.